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Completability of Unimodular Rows in Pure Projective Modules: A Local-Global Perspective

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Abstract

In this paper, we extend classical results on unimodular elements and the completability of unimodular rows in torsion-free pure projective modules over commutative Noetherian rings. We establish a stable freeness criterion that characterises global completability in terms of module-theoretic properties, generalising Serre-Plumstead-type results for a broader class of modules that are not necessarily free. Applications to linear codes over rings are investigated, showing strong structural and distance properties. This work generalises the local-global principle for row completability beyond projective modules and provides a new framework for understanding module decompositions in the algebraic case. Applications include the construction of generator matrices in coding theory over rings, where unimodular rows encode critical structural and distance properties.

Keywords: pure projective module, unimodular row, torsion-free module, stably free module, localisation
Mathematics Subject Classification: 13C12, 13C10, 13F20, 94B05

1 Introduction

In commutative algebra, unimodular elements and their completeability have long been central ideas with extensive applications in algebraic geometry, module theory, and K-theory. Classical results, such as the Quillen-Suslin theorem [11], [13] and the Serre-Plumstead theorems [9] highlight the critical role of unimodular rows in the structure of projective modules and in the stabilisation concept in algebraic K-groups. The work in [1] presents deep results on projective modules, unimodular rows, and their splitting behaviour over affine and polynomial extensions of Noetherian rings. It develops obstruction-theoretic methods and local-global techniques for splitting projective modules and completing unimodular rows, which parallels our focus on completability in pure projective modules. Given a row $u = (p_1, \ldots, p_n) \in P^n$, where P is a module over a commutative ring R, one asks: under what conditions can this row be extended to a full basis of P^n? In free modules, this corresponds to completing u to a matrix in $GL_n(R)$. When P is not necessarily free, the problem becomes more subtle, combining the module's local and homological characteristics. The classic local-global principle says: M is a finitely generated R-module, then $M = 0$ if and only if $M_p = 0$ for every prime ideal $p \in R$ (see [8]). This principle allows to test something globally by checking it locally, at each prime localisation R_p. In the context of module theory, a local-global principle may ask: if a module is free or projective after localisation at all primes, is it globally free or projective? This was precisely the nature of Serre's conjecture, which was resolved affirmatively by Quillen and Suslin in 1976: every finitely generated projective module over a polynomial ring with coefficients in a field is free (see [11], [13]). In equivalent terms, every unimodular row over such rings is globally completable. The local-global principle has applications in various areas, including the study of vector bundles on projective varieties and the classification of projective modules. In [5] the local-global principle, as stated in Lemma 3.11, is applied in the proof of Proposition 3.13, which discusses the transitivity of ETrans(Q, I) on Um(Q, I) for a projective $R[X]$-module P extended from R. Specifically, the local-global principle allows the proof to focus on the case where R is local without loss of generality. Recent literature has expanded on the applications of the local-global principle. In Eisenbud's exposition [2], the principle is implicitly present in the correspondence between modules and sheaves, where global properties of sheaves are determined by local stalks. In Sainhery's thesis [12], the principle underlies the use of cohomology to extend local sections to global ones, illustrating how local exactness informs global behaviour on projective schemes. Despite these advances, the case where module P is not free but satisfies weaker structural conditions, such as being torsion-free and pure projective, remains less understood. Pure projective modules form a natural generalisation of projective modules and provide a rich environment for exploring completability, decomposability, and lifting properties (see [10], [15]). Some articles explore related concepts within the broader context of module theory and homological algebra. An article (see [14]) on representation theory classifies indecomposable modules and examines tensor products, offering a foundation in module structure. Another work by Q. Guo [4] explores projective modules in the context of projective resolutions and modules over Milnor squares, potentially relevant to understanding projective modules. Pure projective modules extend the notion of projective modules by allowing decompositions as direct summands of direct sums of finitely presented modules.

In this article, we introduce and formalise the notions of locally completable and globally completable unimodular rows in P^n, and we establish a result providing a precise local-global criterion for completability of unimodular rows in modules beyond the projective case. We introduce a stable freeness criterion for global completability and provide illustrative examples distinguishing local from global completability. We also include some applications in algebraic coding theory over rings, where unimodular generator matrices are central to the construction of efficient linear codes.

The rest of the article is organised as follows. Section 2 is devoted to the necessary preliminaries required for the article. In section 3 we prove our main result on the local-global principle for completability of unimodular rows. Applications to coding theory are discussed in section 4. Finally, section 5 concludes the paper by summarizing the main results and outlining potential directions for future research.

2 Preliminaries

In this section, we recall basic definitions and concepts that will be used throughout the paper. Unless stated otherwise, all rings considered are commutative Noetherian rings with unity.

Definition 2.1 (Pure Exact Sequence). *[6] An exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of R-modules is defined as pure exact if, for every R-module M', the sequence*

$$0 \to M_1 \otimes_R M' \to M_2 \otimes_R M' \to M_3 \otimes_R M' \to 0$$

is exact again.

Definition 2.2 (Finitely Presented Module). *[3] An R-module M_1 is called a finitely presented module ($\mathcal{FP}$-module) if there exists an exact sequence $0 \to M_2 \to F \to M_1 \to 0$ with F (free) and M_2 finitely generated.*

Definition 2.3 (Pure Projective Module). *[10] An R-module P is said to be pure projective if for every pure exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

The induced sequence

$$0 \longrightarrow \mathrm{Hom}_R(P, A) \longrightarrow \mathrm{Hom}_R(P, B) \longrightarrow \mathrm{Hom}_R(P, C) \longrightarrow 0$$

is exact.

Definition 2.4 (Torsion-Free Module). *[8] An R-module M is called torsion-free if for every nonzero $r \in R$ and $m \in M$, $rm = 0$ implies $m = 0$.*

Definition 2.5 (Unimodular Row)**.** *[8] A row $(p_1, \dots, p_n) \in M^n$ is said to be unimodular if there exist elements $r_1, \dots, r_n \in R$ such that*

$$r_1 p_1 + \cdots + r_n p_n = 1.$$

Definition 2.6 (Completable Unimodular Row)**.** *[8] A unimodular row $(p_1, \dots, p_n) \in M^n$ is said to be completable if it can be extended to the first row of an invertible matrix in $GL_n(R)$.*

3 Global and Local Completability of Unimodular Rows

The study of unimodular rows in module theory often requires distinguishing between local and global properties. While local completability ensures that a unimodular row extends to a basis after localisation at every prime ideal, global completability requires a deeper structural condition. Understanding when local completability guarantees global completability is crucial. The terms locally completable and globally completable are new formal language for existing local-global ideas, which we define as:

Locally Completable Unimodular Row: Let R be a commutative ring, and let P be an R-module. A unimodular row $(p_1, \dots, p_n) \in P^n$ is said to be locally completable if, for every prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$, the localisation $(p_1, \dots, p_n)_{\mathfrak{p}} \in P_{\mathfrak{p}}^n$ extends to an invertible matrix over $R_{\mathfrak{p}}$.

Globally Completable Unimodular Row: A unimodular row $(p_1, \dots, p_n) \in P^n$ is said to be globally completable if there exists a matrix

$$M = \begin{bmatrix} p_1 & \cdots & p_n \\ * & \cdots & * \\ \vdots & & \vdots \\ * & \cdots & * \end{bmatrix} \in \mathrm{GL}_n(R),$$

such that the first row of M is $(p_1, \dots, p_n)$ and the remaining rows complete it to an invertible matrix.

The local-global principle may or may not hold for unimodular rows depending on the ring R and the structure of the module over R. The next example shows that unimodular rows may be locally completable everywhere but still not globally completable.

Example 3.1. *Let $R = k[x, y]_{(x,y)}$ denote the localisation of the polynomial ring $k[x, y]$ at its maximal ideal (x, y), where k is a field.*

Consider the unimodular row

$$u = (x, y) \in R^2.$$

R being a local ring ensures that every unimodular row over it is locally completable. In fact, by definition, every unimodular row over a local ring can

be completed to an invertible matrix. Thus, the row u is locally completable at R.

However, globally over $k[x, y]$ (before localisation), the row (x, y) is not completable to an invertible 2×2 matrix over $k[x, y]$. This is because the ideal generated by x and y is not principal in $k[x, y]$, so no elementary transformations can produce an invertible matrix whose first row is (x, y).

Thus, (x, y) is locally completable at R, but not globally completable over $k[x, y]$.

In the next theorem, we establish a global completeness criterion for unimodular rows in pure projective modules.

Theorem 3.2. *Let R be a commutative Noetherian ring, and let $P \subseteq R^n$ be a finitely generated torsion-free pure projective R-module. Assume that every unimodular row in P^n is locally completable, in the sense that for each prime ideal $\mathfrak{p}$, the localised row $(p_1, \ldots, p_n)_{\mathfrak{p}}$ extends to a basis of $P^n_{\mathfrak{p}}$. Then the following conditions are equivalent:*

(i) Every unimodular row in P^n is globally completable;

(ii) P is stably free.

Proof. Suppose every unimodular row in P^n is globally completable.

Let $u = (p_1, \ldots, p_n) \in P^n$ be a unimodular row. By assumption, there exists an invertible matrix $M \in GL_n(R)$ whose first row is u. Thus, the row u generates a direct summand isomorphic to R inside P^n.

Since P is torsion-free and pure projective, and $P \subseteq R^n$, we can successively choose unimodular rows orthogonal to previously chosen free summands. Proceeding inductively, at each step we split off a copy of R corresponding to a unimodular row.

After finitely many steps (since P is finitely generated), we obtain a decomposition:

$$P \oplus R^m \cong R^{n+m}$$

for some $m \geq 0$. Therefore, P is stably free.

Conversely, suppose P is stably free; that is, there exists $m \geq 0$ such that

$$P \oplus R^m \cong R^{n+m}.$$

Let $u = (p_1, \ldots, p_n) \in P^n$ be a unimodular row.

Because of stable isomorphism, u can be observed within R^{n+m} (identifying P as a direct summand). By the Quillen–Suslin theorem, every unimodular row over R^{n+m} is completable to an invertible matrix.

Thus, the unimodular row u extends to a basis of R^{n+m}. Lifting the result to P, we obtain that u extends to a basis of P^n itself, providing a global completion.

Hence, every unimodular row in P^n is globally completable.

Corollary 3.3. *Let P be a torsion-free pure projective $R[x]$-module of rank greater than $\dim(R)$. Then every unimodular row in P^n is globally completable.*

Proof. By the generalised Plumstead-type theorem under the rank condition, P contains a unimodular element and is stably free. Applying Theorem 3.2, we conclude that every unimodular row in P^n is globally completable. □

Remark 3.4. *Theorem 3.2 highlights the critical role of stable freeness in bridging local and global properties of unimodular rows in pure projective modules, reflecting a local-global principle similar to those in projective module theory and algebraic K-theory.*

4 Applications to Linear Codes

Linear codes over rings have been studied extensively as a generalisation of classical coding theory over finite fields. In this section, we explore the impact of unimodularity and completability on the algebraic and distance properties of linear codes over commutative Noetherian rings.

Theorem 4.1. *Let $C \subseteq R^n$ be a linear code generated by a matrix G whose rows lie in a torsion-free pure projective R-module P. Then the statements below are equivalent:*

(i) C is torsion-free.

(ii) C is pure projective.

(iii) The rows of G form unimodular rows in P^n.

(iv) C is a free R-module.

(v) The minimum Hamming distance $d(C) \geq 2$, unless $C = R^n$.

Proof. $(i) \Rightarrow (ii)$: Over commutative Noetherian rings, torsion-free submodules of pure projective modules inherit purity and are themselves pure projective.

$(ii) \Rightarrow (iii)$: If C is pure projective and finitely generated, it must be a direct summand. Thus, the generator rows can be extended to unimodular rows.

$(iii) \Rightarrow (iv)$: Rows that are unimodular generate a direct summand isomorphic to a free module.

$(iv) \Rightarrow (v)$: Freeness implies that codewords are linearly independent, guaranteeing $d(C) \geq 2$ unless $C = R^n$.

$(v) \Rightarrow (i)$: Minimum distance ≥ 2 ensures no torsion elements in the code.

Remark 4.2. *The equivalence established in Theorem 4.1 extends the previous partial implications shown in our earlier work [7]. Here, unimodularity and global completability guarantee freeness and torsion-freeness of the codes.*

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5 Conclusions

In this paper, we have extended classical results on the existence and completability of unimodular elements in torsion-free pure projective modules over commutative Noetherian rings. We established a kind of local-global principle for the completion of unimodular rows in pure projective modules using stable freeness. The integration of stably free module conditions, local-global principles, and coding-theoretic implications reflects a significant advancement in both structural module theory and its computational applications. This work also lays a foundation for further exploration of module decompositions and completion problems in non-classical settings. Future research can be done to explore a rich interplay between algebra, geometry, and computational methods, pointing towards a deeper theory of modules and codes over general rings.

Author contributions

Conceptualisation: Pooja Gupta, Ratnesh Kumar Mishra; Methodology: Pooja Gupta, Ratnesh Kumar Mishra; Formal analysis and investigation: Pooja Gupta; Writing- original draft preparation: Pooja Gupta; Writing review and editing: Ratnesh Kumar Mishra; Funding acquisition: Ratnesh Kumar Mishra; Supervision: Ratnesh Kumar Mishra.

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Conflict of interest

The authors declare that they have no conflict of interest.

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